math323-f21 (Eppolito)

# Handout: Spooky Reparameterization

31 October 2021

## Introduction

I prepared this document to address a mistake I made in lecture on 29 October 2021. While solving the problem below (taken from an old version of your textbook), a student asked me to compute the integral in another integration order. The results of these different iterated integrals should match because they both compute $\iiint_R f\, dV$.

I couldn't make the computations match during the lecture, and I assumed that I made a mistake somewhere in the arithmetic. However, the true mistake was more sinister: I trusted that the description of the region from the text was a parameterization—that is not the case here. A full explanation of the mistake is given below.[1]

## Problem

The problem I gave in lecture is reproduced below.

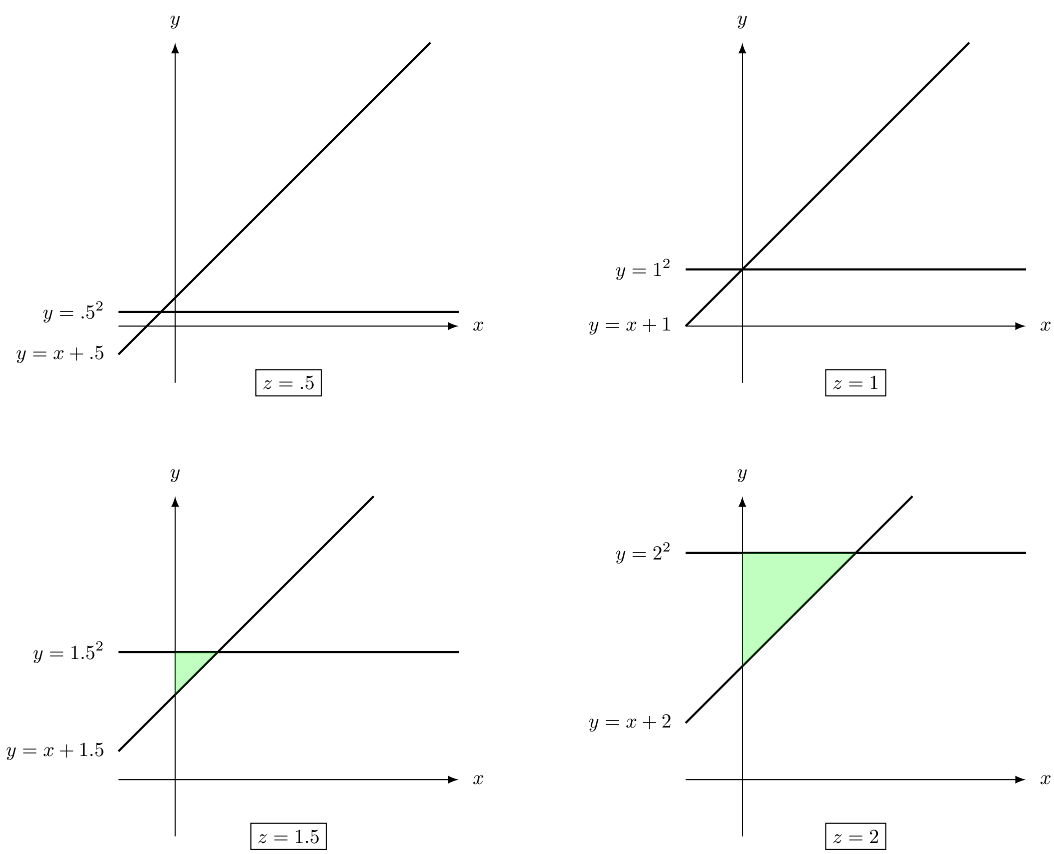
1. Integrate $2x - y$ over $R = \{(x, y, z) : 0 \le z \le 2, 0 \le y \le z^2, 0 \le x \le y - z\}$.

## Mistake

Consider a $z = k$ cross section of the given region. Fixing a constant $0 \le k \le 2$, the cross section $R_k = \{(x, y, k) : 0 \le y \le k^2, 0 \le x \le y - k\}$. Because $k$ was fixed, both $y = k^2$ and $x = y - k$ are lines.

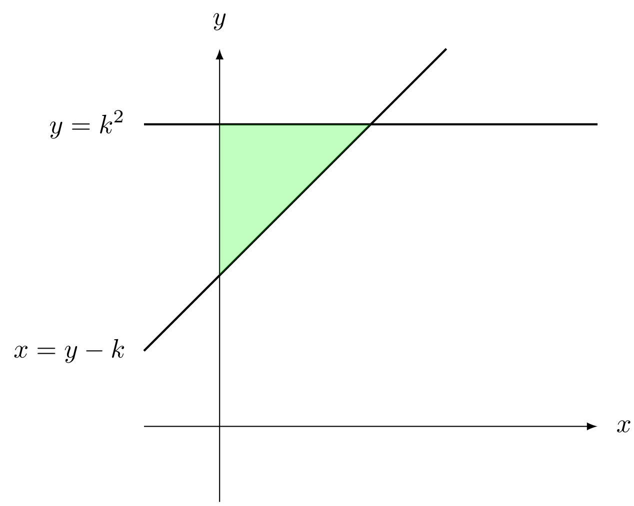
I drew a picture in the lecture when I reparameterized the region, but I was too tired to notice that this picture tells us that the original description of the region is not optimal (and hence not good for building an iterated integral). In particular, notice that the true $y$-bounds in the image above are $k \le y \le z^2$. Thus a better (though equivalent) description of the $z = k$ cross section is $\{(x, y, k) : k \le y \le k^2, 0 \le x \le y - k\}$. Indeed, the condition $k \le y$ is also implied by the last inequality: if $0 \le x \le y - k$, then $k \le x + k \le y$ simply by adding $k$ to the inequality.

We also ought to check the $z$-bounds; there are further implicit bounds on $k$. Because $k \le y \le k^2$, we see that $0 \le y - k \le k^2 - k$, and thus $k \ge 1$ by $k \ge 0$; below are a few pictures to convince you of this.




[1] If you email me with subject line "never trust a maths book", I will give you a full quiz mark.

We thus have the following picture generically (scale omitted purposely); this is the picture I drew in lecture.




The true parameterization of this region is $R = \left\{(x, y, z) : 1 \leq z \leq 2, z \leq y \leq z^2, 0 \leq x \leq y - z\right\}$ by our work above. The remainder of this document is devoted to alternate solutions computing the integral.

**Integration in $dx\ dy\ dz$-Order**

**Solution:** We compute the integral as follows (the parameterization we computed is already good for this order).

$$\begin{aligned}
\iiint_R (2x - y)\ dV &= \int_{z=1}^{2} \int_{y=z}^{z^2} \int_{x=0}^{y-z} (2x - y)\ dx\ dy\ dz \\
&= \int_{z=1}^{2} \int_{y=z}^{z^2} \left[x^2 - xy\right]_{x=0}^{y-z}\ dy\ dz \\
&= \int_{z=1}^{2} \int_{y=z}^{z^2} ((y-z)^2 - (y-z)y - 0)\ dy\ dz \\
&= \int_{z=1}^{2} \int_{y=z}^{z^2} -z(y-z)\ dy\ dz \\
&= \int_{z=1}^{2} -z\left[\tfrac{1}{2}y^2 - yz\right]_{y=z}^{z^2}\ dz \\
&= \int_{z=1}^{2} -z((\tfrac{1}{2}(z^2)^2 - z^2 z) - (\tfrac{1}{2}z^2 - zz))\ dz \\
&= \int_{z=1}^{2} -z(\tfrac{1}{2}z^4 - z^3 - \tfrac{1}{2}z^2 + z^2)\ dz \\
&= \int_{z=1}^{2} (-\tfrac{1}{2}z^3 + z^4 - \tfrac{1}{2}z^5)\ dz \\
&= \left[-\tfrac{1}{8}z^4 + \tfrac{1}{5}z^5 - \tfrac{1}{12}z^6\right]_{z=1}^{2} \\
&= (-2 + \tfrac{32}{5} - \tfrac{16}{3}) - (-\tfrac{1}{8} + \tfrac{1}{5} - \tfrac{1}{12}) \\
&= -2 \cdot \tfrac{120}{120} + \tfrac{31 \cdot 24}{5 \cdot 24} - \tfrac{16 \cdot 40}{3 \cdot 40} + \tfrac{1 \cdot 15}{8 \cdot 15} + \tfrac{1 \cdot 10}{12 \cdot 10} \\
&= \tfrac{-240+744-640+15+10}{120} \\
&= -\tfrac{111}{120} = -\tfrac{37}{40}
\end{aligned}$$

**Integration in $dy\ dx\ dz$-Order**

**Solution:** To reparameterize $R$ we again consider the $z = k$ cross section (pictured above); this yields a reparameterization $R = \left\{(x, y, z) : 1 \le z \le 2, 0 \le x \le z^2 - z, x + z \le y \le z^2\right\}$. We now compute the iterated integral.

$$
\begin{aligned}
\iiint_R (2x - y)\ dV &= \int_{z=1}^{2} \int_{x=0}^{z^2-z} \int_{y=x+z}^{z^2} (2x - y)\ dy\ dx\ dz \\
&= \int_{z=1}^{2} \int_{x=0}^{z^2-z} \left[2xy - \tfrac{1}{2}y^2\right]_{y=x+z}^{z^2}\ dx\ dz \\
&= \int_{z=1}^{2} \int_{x=0}^{z^2-z} ((2xz^2 - \tfrac{1}{2}(z^2)^2) - (2x(x+z) - \tfrac{1}{2}(x+z)^2))\ dx\ dz \\
&= \int_{z=1}^{2} \int_{x=0}^{z^2-z} (2xz^2 - \tfrac{1}{2}z^4 - 2x^2 - 2xz + \tfrac{1}{2}x^2 + xz + \tfrac{1}{2}z^2)\ dx\ dz \\
&= \int_{z=1}^{2} \int_{x=0}^{z^2-z} (2xz^2 - \tfrac{1}{2}z^4 - xz - \tfrac{3}{2}x^2 + \tfrac{1}{2}z^2)\ dx\ dz \\
&= \int_{z=1}^{2} \left[x^2z^2 - \tfrac{1}{2}xz^4 - \tfrac{1}{2}x^2z - \tfrac{1}{2}x^3 + \tfrac{1}{2}xz^2\right]_{x=0}^{z^2-z}\ dz \\
&= \int_{z=1}^{2} ((z^2-z)^2z^2 - \tfrac{1}{2}(z^2-z)z^4 - \tfrac{1}{2}(z^2-z)^2z - \tfrac{1}{2}(z^2-z)^3 + \tfrac{1}{2}(z^2-z)z^2 - 0)\ dz \\
&= \int_{z=1}^{2} (z^4(z-1)^2 - \tfrac{1}{2}z^5(z-1) - \tfrac{1}{2}z^3(z-1)^2 - \tfrac{1}{2}z^3(z-1)^3 + \tfrac{1}{2}z^3(z-1))\ dz \\
&= \int_{z=1}^{2} z^3(z-1)(z(z-1) - \tfrac{1}{2}z^2 - \tfrac{1}{2}(z-1) - \tfrac{1}{2}(z-1)^2 + \tfrac{1}{2})\ dz \\
&= \int_{z=1}^{2} z^3(z-1)(z(z-1) - \tfrac{1}{2}(z^2-1) - \tfrac{1}{2}(z-1) - \tfrac{1}{2}(z-1)^2)\ dz \\
&= \int_{z=1}^{2} z^3(z-1)^2(z - \tfrac{1}{2}(z+1) - \tfrac{1}{2} - \tfrac{1}{2}(z-1))\ dz \\
&= \int_{z=1}^{2} z^3(z-1)^2(z - \tfrac{1}{2}(z+1+1+z-1))\ dz \\
&= \int_{z=1}^{2} z^3(z-1)^2(z - z - \tfrac{1}{2})\ dz \\
&= \int_{z=1}^{2} -\tfrac{1}{2}z^3(z-1)^2\ dz \\
&= \int_{z=1}^{2} (-\tfrac{1}{2}z^3 + z^4 - \tfrac{1}{2}z^5)\ dz = -\tfrac{37}{40}
\end{aligned}
$$